Eur. Phys. J. C 55, 683–686 (2008)
DOI 10.1140/epjc/s10052-008-0621-7

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article – Theoretical Physics

On the distances between entangled pseudoscalar mesons states

M. Genovese[a]

Istituto Nazionale di Ricerca Metrologica, Strada delle Cacce 91, 10135 Torino, Italy

Received: 7 February 2008 / Revised version: 15 April 2008 /
Published online: 17 May 2008 –

Abstract. Entangled states of pseudoscalar mesons represent a very interesting tool for studying the foundations of quantum mechanics, e.g. for testing the Bell inequalities. Recently, they also emerged as a test bench for quantum information protocols. On the other hand, from a quantum information point of view, the characterization of the distance between two quantum states is a topic of the utmost importance. In this work, with the purpose of providing a useful tool for further investigations, we address the problem of which distance allows for a better discrimination between density matrices appearing in pseudoscalar phenomenology.

PACS. 13.20.Eb; 03.65.Bz; 03.65.Ta; 03.67.-a

1 Introduction

Entangled states of neutral pseudoscalar mesons (K_0 and B_0) represent a very interesting tool for studying the foundations of quantum mechanics [1], e.g. as a test of the Bell inequalities [2–33]. Recently, they also emerged as a test bench for quantum information protocols [34]: for example, protocols for quantum teleportation [35] and a quantum eraser [36, 37] have been proposed. These new proposals represent a very interesting innovative application for Φ and B factories: indeed not only to realize similar experiments with different physical systems is of large interest, but also the use of mesons discloses new possibilities [1–35].

On the other hand, from a quantum information point of view, a topic of the utmost importance is the characterization of the distance between two quantum states [39], that, beyond its own intrinsic interest, is needed, for example, for assessing teleportation [40], purification [41], quantum cloning [42], remote state preparation [43] and state estimation [44]. The use of the neutral pseudoscalars in this context would represent a further interesting application to quantum information. Furthermore, the notion of the distance between quantum states for kaons is also a useful tool when studying the developing of mixed states from pure ones in the presence of (quantum gravity induced) decoherence [45–47] or when searching for CPT violation [48, 49].

The problem of which distance is more convenient for distinguishing different quantum states is not solved in general, but only specific cases, like single qubits in a noisy channel [39], have been discussed.

In this work we address this problem in neutral pseudoscalar meson phenomenology. Several different experimental cases that can be implemented, which could represent an interesting "arena" for future studies, are investigated. The results presented provide a useful tool for future research addressed to study of the application of mesons both to quantum information and decoherence, suggesting which distance is better to use in specific cases.

2 Distances between entangled pseudoscalar mesons states

In recent years various distances among quantum states have been defined [38] with the purpose of comparing the states and eventually defining an entanglement measure.

Here we will consider three of the most used distances [38]: the Bures, the Hilbert–Schmidt and the trace distances.[1]

The Bures distance,

$$D_{\mathrm{B}}(\rho,\sigma)=\sqrt{1-\left(\mathrm{Tr}\left[\sqrt{(\sqrt{\sigma}\rho\sqrt{\sigma})}\right]\right)^2}\,, \tag{1}$$

represents the shortest path connecting two fibers (in the Hilbert–Schmidt fiber bundle) lying over the two density matrices. It is both Riemannian and monotone [38].

The Hilbert–Schmidt distance,

$$D_{\mathrm{HS}}(\rho,\sigma)=\left(\frac{\sqrt{\mathrm{Tr}[(\rho-\sigma)\cdot(\rho-\sigma)]}}{\sqrt{2}}\right), \tag{2}$$

[a] e-mail: m.genovese@inrim.it

[1] In the following we use normalizations as in [39].

is the Euclidean distance deriving from the definition of a scalar product in the Hilbert–Schmidt space. It is Riemannian, but not monotone.

Finally, the trace distance,

$$D_{\mathrm{tr}}(\rho,\sigma)=\frac{1}{2}\,\mathrm{Tr}\,|\rho-\sigma|\,, \tag{3}$$

simply derives by the definition of norm. It is not Riemannian, but it is monotone. It coincides with the Hilbert–Schmidt one for single qubits.

In the following we will consider the basis $K_{\mathrm{L}}=\{1,0\}$, $K_{\mathrm{S}}=\{0,1\}$, K_{L} and K_{S} being the long and short living states of K_0, respectively (a similar discussion, mutatis mutandis, for mass and beauty eigenstates can be given for B mesons as well [31]). The experimental determination of the meson states can be achieved through the decays [2–30].

The density matrix for the singlet state[2]

$$|\Psi^-\rangle=\frac{|K^0\rangle|\bar{K}^0\rangle-|\bar{K}^0\rangle|K^0\rangle}{\sqrt{2}}=\frac{|K_{\mathrm{L}}\rangle|K_{\mathrm{S}}\rangle-|K_{\mathrm{S}}\rangle|K_{\mathrm{L}}\rangle}{\sqrt{2}}\,, \tag{4}$$

typically produced at Φ factories, is therefore

$$\varrho_S=\begin{pmatrix}0&0&0&0\\0&\frac{1}{2}&-\frac{1}{2}&0\\0&-\frac{1}{2}&\frac{1}{2}&0\\0&0&0&0\end{pmatrix}. \tag{5}$$

As a first example, mathematically almost trivial, let us consider the distances between the singlet and the transformed state after one of the two components has been regenerated in a slab [50], which is an interesting effect for a simple experimental implementation.

The transformation is [32] (f complex, with typical values per unit thickness $|f|\sim 10^{-3}\,\mathrm{mm}^{-1}$ [32, 33])

$$U=\frac{1}{\sqrt{(1+|f|^2)}}\begin{pmatrix}1&f&0&0\\f&1&0&0\\0&0&1&f\\0&0&f&1\end{pmatrix}. \tag{6}$$

In this case the three distances coincide:

$$\begin{aligned}D_{\mathrm{B}}(\varrho_S,U\varrho_S U^\dagger)&=D_{\mathrm{tr}}(\varrho_S,U\varrho_S U^\dagger)\\&=D_{\mathrm{HS}}(\varrho_S,U\varrho_S U^\dagger)=\sqrt{\frac{|f|^2}{1+|f|^2}}\,.\end{aligned} \tag{7}$$

Let us now consider the more interesting case of the decoherence effect on a singlet [51], t being the time:

$$\begin{pmatrix}0&0&0&0\\0&\frac{1}{2}&-\frac{1}{2}\mathrm{e}^{-lt}&0\\0&-\frac{1}{2}\mathrm{e}^{-lt}&\frac{1}{2}&0\\0&0&0&0\end{pmatrix}. \tag{8}$$

This density matrix can describe both decoherence effects due to interaction with the environment [51] and to gravity induced decoherence [45–47]. Thus, one is interested in distinguishing the decohered state from the unaffected singlet: a point that has a large interest for future experiments.

Let us define $\tau=1/l$ the characteristic decoherence time scale. Again the trace and Hilbert–Schmidt distances coincide. If one looks at Fig. 1, one can observe that the Bures distance allows for a better discrimination from the unaffected singlet at times smaller than $\tau_s=0.69\tau$, since it has a larger gradient. On the other hand, the Hilbert–Schmidt one is more sensitive at larger values of time. τ_s shifts toward smaller values if a background is added, $\varsigma_D=x\sigma_D+(1-x)\mathbf{1}/4$, where $\tau_s=0.67\tau$ for a 1% background, going to 0.51 for a 10% one.

As a further example, one can consider the statistical mixture of the singlet with a singlet regenerated by a slab, $\varrho=x\varrho_S+(1-x)U\varrho_S U^\dagger$, whose distance is measured from the singlet. In this case the Hilbert–Schmidt distance, coinciding with the trace one, is always less sensitive than the Bures one.

The general case of the distance between the regenerated singlet $U(f1)\varrho_S U(f1)^\dagger$ and the mixing of the singlet regenerated by two different slabs, $xU(f1)\varrho_S U(f1)^\dagger+(1-x)U(f2)\varrho_S U(f2)^\dagger$ represents an interesting example, where a more complex situation can be studied (and that eventually can also be implemented experimentally). This case must be considered with a full variation of modulus and phase of $f1$ and $f2$. An example is reported in Fig. 2, where one can observe the plot of the difference between Bures and Hilbert–Schmidt distances when $f1$ and $f2$ are real and $x=0.5$. For this case one can see that, when varying $|f2|$ at fixed $|f1|$, the Hilbert–Schmidt distance is more sensitive than the Bures one when $|f2|<|f1|$ and vice versa. When a phase is added to $f1$, the situation does not change substantially, except for a decrease of the difference $D_{\mathrm{B}}-D_{\mathrm{HS}}$ at small $f2$ (vanishing when the phase goes to $\pi/2$) and a small decrease of the inversion point under $|f1|$. On the other hand, when we scan the variation with x at fixed values of $f1$ and $f2$, the Bures distance is always more

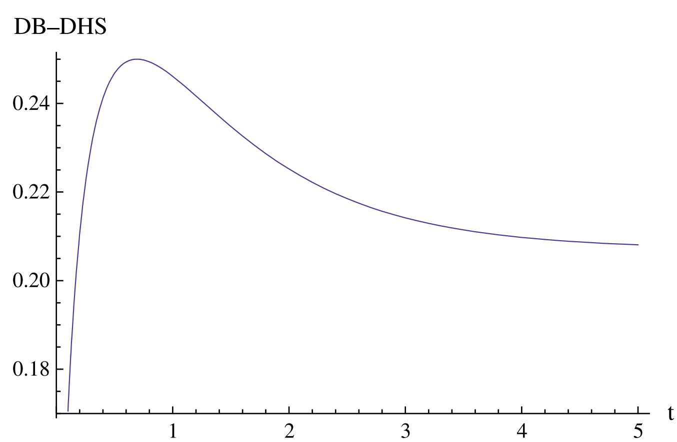

Fig. 1. Difference between the Bures and Hilbert–Schmidt distances between singlet and decohered singlet as a function of time (measured in units corresponding to the decoherence characteristic time $1/l$)

[2] In the following, the very small CP violation effects will be neglected, since they do not substantially affect the presented results.

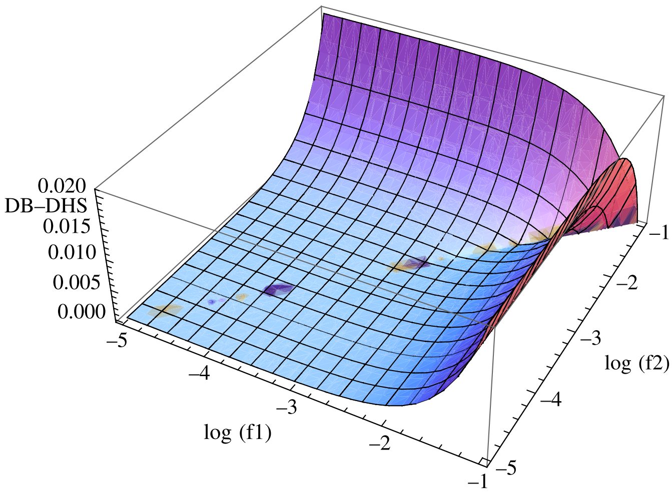

Fig. 2. Three-dimensional plot of the difference between the Bures and Hilbert–Schmidt distances between $U(f1)\varrho_S U(f1)^\dagger$ and $(U(f1)\varrho_S U(f1)^\dagger + U(f2)\varrho_S U(f2)^\dagger)/2$ as a function of $\log(f1)$ and $\log(f2)$

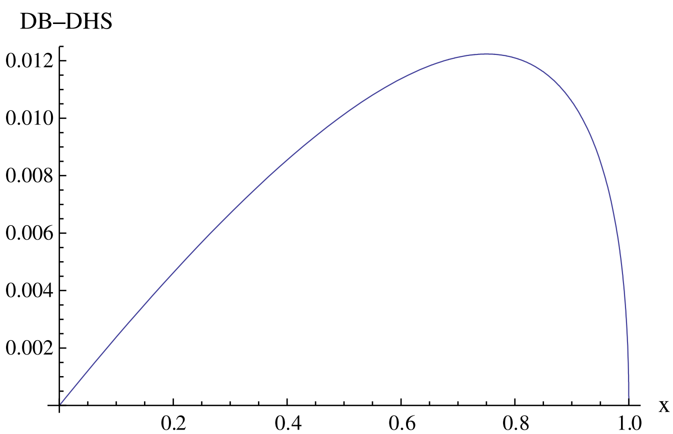

Fig. 3. Difference between the Bures and Hilbert–Schmidt distances between $U(f1)\varrho_S U(f1)^\dagger$ and $xU(f1)\varrho_S U(f1)^\dagger + (1-x)U(f2)\varrho_S U(f2)^\dagger$, $f1 = 0.05$, $f2 = 0.001$, as a function of x

sensitive than the Hilbert–Schmidt one when $x < 0.75$ and vice versa for $x > 0.75$ (e.g. see Fig. 3).

In conclusion, we consider the case where the singlet is mixed with a background, $\varrho = x\varrho_S + (1-x)\mathbf{1}/4$ (in quantum information terminology a "depolarizing channel"). Here, the Hilbert–Schmidt distance is always more sensitive than the trace one. On the other hand, it is less sensitive than the Bures distance for small values of x ($x < 0.5$); see Fig. 4. Finally, the Bures distance is more discriminant than the trace one up to $x = 2/3$ and less for larger x. The situation remains exactly the same when considering the mixing $U\varrho_S U^\dagger + (1-x)\mathbf{1}/4$ between regenerated singlet and background and changing the phase and the modulus of f (in a reasonable interval corresponding to the experimental accessible zone[3]).

[3] That is, $|f| < 0.1$. For $|f|$ approaching 1 the Hilbert–Schmidt distance becomes more sensitive also at smaller x.

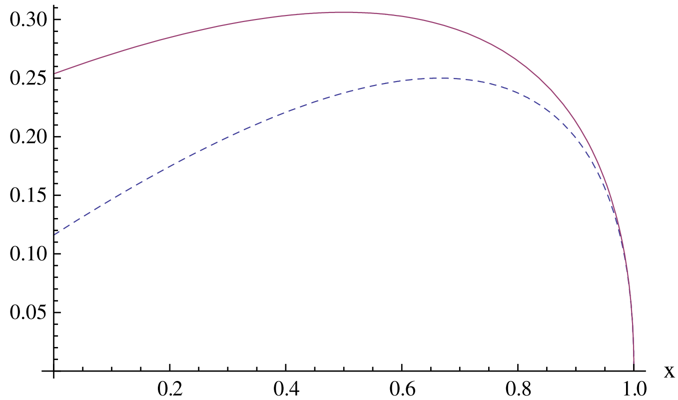

Fig. 4. Difference between the Bures and Hilbert–Schmidt distances (*solid*) and between the Bures and Trace distances (*dashed*) between singlet and singlet mixed with background, $\varrho = x\varrho_S + (1-x)\mathbf{1}/4$, as a function of x

In summary, our calculations show that the choice of which distance to consider for studying the 0^{-+} mesons entanglement depends on the cases, in particular the Hilbert–Schmidt and the Bures ones look most promising. Our results suggest which would be the best selection for some specific different situations of theoretical and experimental interest.

3 Conclusions

In this work we have presented a study addressed to an estimate of which distance between states is more sensitive when comparing different density matrices that can be met with in neutral 0^{-+} meson phenomenology.

This is a relevant problem since neutral pseudoscalar mesons represent an interesting test bench of quantum information protocols, alternative to more traditional ones based on photons and/or atoms. Many different interesting cases, from a quantum information point of view, can be experimentally achieved by exploiting the regeneration phenomenon. Furthermore, our results can find application when studying the developing of mixing from a pure state due to gravitational induced (or not) decoherence.

Albeit that our results are far from exhaustive, nevertheless they include many interesting examples pointing out how the sensitivity of different distances varies from case to case and must be kept carefully into account when applied to neutral pseudoscalar phenomenology. In general, they suggest that Hilbert–Schmidt and Bures distances are the most promising to be considered and they represent a possible guide for a choice related to specific physical examples, that could be of use in further studies addressed to plan meson experiments both on quantum information and decoherence.

Acknowledgements. This work has been supported by MIUR (PRIN 2005023443-002) and by Regione Piemonte (E14).

References

1. M. Genovese, Phys. Rep. **413**, 319 (2005)
2. F. Uchiyama, Phys. Lett. A **231**, 295 (1997)
3. R.A. Bertlmann, W. Grimus, B.C. Hiesmayr, Phys. Lett. A **289**, 21 (2001)
4. F. Benatti, R. Floreanini, Phys. Rev. D **57**, R1332 (1998)
5. F. Benatti, R. Floreanini, Eur. Phys. J. C **13**, 267 (2000)
6. M. Genovese, Eur. Phys. J. C **42**, 25 (2005)
7. R.A. Bertlmann, W. Grimus, Phys. Lett. B **392**, 426 (1997)
8. R.A. Bertlmann, W. Grimus, Phys. Rev. D **58**, 034 014 (1998)
9. G.C. Ghirardi et al., The DAΦNE Physical Handbook, ed. by L. Maiani, G. Pancheri, N. Paver (INFN, Frascati, 1992), Vol. I
10. J. Six, Phys. Lett. B **114**, 200 (1982)
11. P.H. Eberhard, Nucl. Phys. B **398**, 155 (1993)
12. B. Ancochea, Phys. Rev. D **60**, 094 008 (1999)
13. N. Gisin, A. Go, Am. J. Phys. **69**, 264 (2001)
14. B.C. Hiesmayr, Found. Phys. Lett. **14**, 231 (2001)
15. R.A. Bertlman et al., Phys. Lett. A **332**, 355 (2004)
16. A. Bramon et al., quant-ph/0410122
17. R.H. Dalitz, G. Garbarino, quant-ph/0011108
18. P. Privitera, F. Selleri, Phys. Lett. B **296**, 261 (1992)
19. F. Selleri, Phys. Rev. A **56**, 3493 (1997)
20. A. Bramon, G. Garbarino, Phys. Rev. Lett. **88**, 040 403 (2002)
21. H. Albrecht, Phys. Lett. B **192**, 245 (1987)
22. M. Genovese et al., Phys. Lett. B **513**, 401 (2001)
23. M. Genovese et al., Found. Phys. Lett. **32**, 589 (2002)
24. M. Genovese, Phys. Rev. A **69**, 022 103 (2004)
25. A. Apostolakis et al., Phys. Lett. B **422**, 339 (1998)
26. A. Go, J. Mod. Opt. **51**, 991 (2004)
27. R. Foadi, F. Selleri, Phys. Rev. A **61**, 012 106 (2000)
28. A. Bramon, G. Garbarino, Phys. Rev. Lett. **89**, 160 401 (2002)
29. A. Go et al., quant-ph/0702267
30. Y.-B. Ding et al., quant-ph/0702271
31. A. Pompili, F. Selleri, Eur. Phys. J. C **14**, 469 (2000)
32. A. Bramon, M. Nowakowski, Phys. Rev. Lett. **83**, 1 (1999)
33. A. Di Domenico, Nucl. Phys. B **450**, 293 (1995)
34. R.A. Bertlmann, B.C. Hiesmayr, Quantum Inf. Proc. **5**, 421 (2006)
35. Y. Shih, Phys. Lett. B **641**, 75 (2006)
36. A. Bramon et al., Phys. Rev. Lett. **92**, 020 405 (2004)
37. A. Bramon et al., Phys. Rev. A **68**, 06 211 (2004)
38. I. Bengtsson, K. Zyczkowski, Geometry of Quantum States (Cambridge University Press, Cambridge, 2006)
39. S. Salvini et al., Laser Phys. **17**, 552 (2007)
40. F. Verstraete, H. Verschelde, Phys. Rev. Lett. **90**, 097 901 (2003)
41. C.H. Bennet et al., Phys. Rev. Lett. **76**, 722 (1996)
42. N. Gisin, S. Massar, Phys. Rev. Lett. **79**, 2153 (1997)
43. A.X. Chen et al., Commun. Theor. Phys. **43**, 631 (2005)
44. E.M.A. Paris, J. Rehacek, Quantum State Estimation (Springer, Berlin, 2004)
45. J. Ellis et al., Nucl. Phys. B **421**, 381 (1984)
46. F. Benatti, R. Floreanini, Phys. Lett. B **401**, 337 (1997)
47. H.-J. Gerber, Phys. Rev. Lett. **80**, 2969 (1998)
48. R.A. Briere, L.H. Orr, Phys. Rev. D **40**, 2269 (1989)
49. J. Ellis et al., Phys. Rev. D **53**, 3846 (1996)
50. E. Leader, E. Predazzi, An Introduction to Gauge Theories and Modern Particle Physics (Cambridge University Press, Cambridge, 1996)
51. R.A. Bertlmann et al., Phys. Rev. A **68**, 012 111 (2003)